AN INVESTIGATION INTO THE EFFECT OF COOLANT FLOW ON THE VIBRATION CHARACTERISTICS OF HOLLOW BLADES CONVEYING FLUID

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ABSTRACT

This paper treats the free vibration of hollow symmetrical turbo-machinery blades conveying cooling fluid. The blade motion is described by using a simplified shell theory, whereas the fluid forces are described by the linearized potential flow theory. Natural frequencies are presented for the axial and circumferential modes and the effect of flow velocity and other parameters are discussed. Two models are constructed and tested with air and water as the flowing fluid. Also a beam approximation is used to justify the results at least in a qualitative manner. Experimental data and theoretical results are in good agreement and they all show that the fluid flow tends to decrease the free vibration-natural frequencies.

NOMENCLATURE

A Area of fluid flow.
\(A_{kj}\) Matrix elements with \(k=1,2,...,16\) and \(j=1,2,...,16\).
b Width of model.
\(C_k\) Constants of integration with \(k=1,2,...,16\).
\(C_s\) Velocity of sound.
\(E\) Modulus of elasticity.
\(I_1\) Complex root \((-1)^{1/2}\).
\(l\) Blade length.
\(M\) Bending moment.
\(m\) Circumferential mode number.
\(N\) Normal stretching force.
\(n\) Axial mode number.
\(Q\) Shear force.
\(R\) Shell radius.
\(r,\theta, t\) Radial, tangential and time coordinate, respectively.
\(U\) Undisturbed fluid flow velocity.
\(u, v, w\) Axial, circumferential and normal deflection of middle surface, respectively.
\(V_r\) Flow velocity in the radial direction.

1. INTRODUCTION

For high engine performance, modern turbines utilize cooling mechanisms for the first stage inlet guide vanes and rotor blades. The cooling is accomplished by feeding gas or liquid through longitudinal holes, cavities or passages provided in the blade structure, see for example the typical blades shown in figures (1) and (2).

Investigation of the dynamic interaction between the cooling fluid and the elastic blade structure was not found in the open literature. However, previous studies on solid-fluid interaction problems were in the field of vibration and stability elements such as fluid conveying pipes \(1,2,3,4\) and plates and panels under the effect of subsonic or supersonic flow of fluid \(5,6\).

A thin blade is needed for higher turbine performance, therefore shell theories should be used since the number in brackets designate References at the end of paper.
Fig. 1 A typical cooled turbo-machinery blade. Conventional beam model.

Fig. 2 A typical cooled turbo-machinery blade. Conventional shell model.

Fig. 3 Symmetrical hollow circular blade

The fluid is supposed to flow axially at a constant rate through the space between the two shell elements. In formulating the mathematical model, small vibrations are only considered. The simplified theory for long shell (1/R) stated by Al-Jumaily and Faulkner (1971) was combined with a linearized potential flow theory (assuming ideal compressible, irrotational and non-heat conducting fluid flow) to give the complete equations describing the motion of each part of the composite blade.

The system equations were solved using a closed functional representation which the simplified shell theory allows for. While the difficulty in obtaining analytical solutions for the velocity potential function was accommodated using an equivalent technique (this will be illustrated in the sequel).

2. ANALYTICAL PROCEDURE

2.1 Governing Equation

Making use of the assumptions leading to the simplified shell theory, Ref. (1971), the equation of motion with the tangential inertias neglected of a shell element exposed to a one side axial flow of fluid may be written as (Appendix 1):

$$\frac{\partial^2\psi}{\partial x^2} + \frac{12}{R} \frac{\partial\psi}{\partial x} + \frac{12}{Eh^3} \frac{\partial^2\psi}{\partial x^2} = 0$$

Assuming potential fluid flow, the dynamic fluid pressure $P_d$ acting on the shell element can be found to be (Appendix 2):

$$P_d = \frac{12}{Eh^3} \frac{\partial^2\psi}{\partial x^2} + \frac{U_d^2}{(1-\nu^2)e} \frac{\partial^2\psi}{\partial x^2} + \frac{U_d^2}{(1-\nu^2)e} \frac{\partial^2\psi}{\partial x^2}$$

Considering the individual shell element of the composite blade shown in Fig. (3), the governing equation of motion of each part can be obtained by combining Eq. (1) and Eq. (2) with the proper sign convention for the direction of the dynamic fluid pressure acting on each part. Accordingly, the equations may be written as:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{12}{R} \frac{\partial\psi}{\partial x} + \frac{12}{Eh^3} \frac{\partial^2\psi}{\partial x^2} = 0$$

$$\frac{12}{Eh^3} \frac{\partial^2\psi}{\partial x^2} = \frac{U_d^2}{(1-\nu^2)e} \frac{\partial^2\psi}{\partial x^2}$$

with the positive sign for the bottom shell element $(j=2)$ and the negative sign for the top shell element $(j=1)$.

2.2 Boundary Conditions

Using the "Matching of Continuous Boundary Conditions Technique", (1971) requires to state the boundary conditions of each element of the composite blade. The blade is assumed to be clamped at $x=0$ and free at the other three edges. The boundary conditions for the individual shell element at the ends $x=0$ and $x=L$ may be written as:

$$u_j = v_j = w_j = \frac{\partial w_j}{\partial x} = 0 \quad \text{at} \quad x=0 \quad \text{at} \quad x=L$$

$$x_j = (x_{s})_j = (x_{t})_j = (x_{d})_j = 0 \quad \text{at} \quad x=0 \quad \text{at} \quad x=L$$

where $u_j, v_j, w_j$ are the displacement components in the $x, y, z$ directions, respectively.
The two shell elements are assumed to be rigidly connected along their straight edges. This can be fulfilled by ensuring the continuity of all deflections, slopes, moments and forces for both shell elements at the junctions \( 0-0 \) and \( 0-8 \). Selecting the coordinates of the top shell (shell 1) as the reference coordinates, Fig. (4), the boundary conditions at the junctions may be written as:

At \( \theta_1 = \theta_2 = 0 \)

1. Kinematic:

\[
\begin{align*}
&w_1 - (w_2 \cos \alpha - v_2 \sin \alpha) = 0 \\
&v_1 - (v_2 \cos \alpha + w_2 \sin \alpha) = 0 \\
&u_1 - u_2 = 0 \\
&\frac{\partial w_1}{\partial \theta} - \frac{\partial w_2}{\partial \theta} = 0
\end{align*}
\]

2. Equilibrium:

\[
\begin{align*}
&N_{11} + N_{12} = 0 \\
&Q_{11} + Q_{12} = 0 \\
&N_{21} + N_{22} = 0 \\
&N_{31} + N_{32} = 0
\end{align*}
\]

where \( \alpha = \frac{\theta_1 - \theta_2}{2} \)

The boundary conditions along the other junctions \( \theta_1 = \theta_01 \) and \( \theta_2 = \theta_02 \) are similar to those given in Eqs. (5) except \( \alpha \) has to be replaced by \( -\alpha \) in all of the equations.

![Fig. 4 Forces and displacement at the boundaries](image)

2.3 Solution

Complete solution for Eqs. (3) subjected to the boundary conditions Eqs. (4) and Eqs. (5) can be obtained by finding analytical expressions for the stress function \( \Phi \) and the ratio \( (\Phi(R)/\bar{R}(R)) \). Solution for the stress function may be assumed to be of the following form:

\[
\Phi_j(f, \theta, z) = \sum_{k=1}^{\infty} (C_k \cdot Y_{mn}(\theta)) e^{\text{iat}}
\]

with

\[
F_j(f, z) = X_n(f), \quad (Y_{mn}(\theta))_j = \text{same}
\]

where \( X_n \) is the beam function which satisfies the boundary conditions at the ends \( f = 0 \) and \( f = 1 \). (Appendix 3) and \( Y_{mn} \) are circumferential mode functions to be determined.

The presence complex term in the coefficient \( B_k \), Eq. (7b) indicates that the system circular frequency includes a complex part which may result either in damping of the system vibrations or in system instability depending on whether it is positive or negative, respectively [9].

Solution of the eight order differential equation (7a) with complex coefficients is difficult to obtain. Special case solution may be obtained when the complex term representing the Coriolis fluid force component in the coefficient \( B_k \) is neglected. This reduces the circular frequency \( \omega \) to a wholly real part giving the system natural frequencies while no idea about the system damping or system instability can be obtained.

With the above simplification the general solution for the \( Y_{mn} \) functions may be written as:

\[
(Y_{mn}(\theta))_j = \sum_{k=1}^{\infty} (C_k \cdot Y_{mn}(\theta))_j
\]

where \( Y_k \)'s are given by table (5) for the two ranges of the characteristic coefficient \( r^2 - k^2 \) and \( C_k \) are constants of integration.

The unknown ratio \( (\Phi(R)/\bar{R}(R)) \) may be found by solving for the velocity potential function \( \Psi \), Eq. (28) with the boundary condition given by Eq. (29). This seems to be cumbersome however an approximate value may be obtained by equating the fluid forces acting on a shell element to that acting on a beam element, i.e. by equating Eq. (32) and Eq. (33). Accordingly the ratio is found to be:

\[
\left( \frac{\Phi(R)}{\bar{R}(R)} \right)_j = \frac{A_j}{b_j}
\]

where \( b_j \) for each element may be taken as the model width, and \( A_j \) is the equivalent fluid flow area which is related to the hollow blade area \( A_f \) by:
\[ A_1 = N \frac{A_f}{r} \quad \text{(10a)} \]

\[ A_2 = (1-N) \frac{A_f}{r} \quad \text{(10b)} \]

where \( N \) is a dynamic added mass coefficient representing the ratio of the fluid affecting part on the upper shell element. The exact value of \( N \) depends on the position of the shell element relative to the fluid, the blade orientation and the fluid flow velocity. However, for the blade in horizontal position the value of \( N \) may be taken to be 0.3, Ref. ID \\]

2.4 Frequency Equation

The four boundary conditions at each of the ends \( x=0 \) and \( x=1 \) or \( \theta = \pi \) are automatically satisfied by using the solution (6) and the only boundary conditions that have to be satisfied are those along the junctions \( \theta = 0 \) and \( \theta = \pi \), Eq. (5). Satisfying these boundary conditions enables the evaluation of the sixteen constants of integration \( C_k \) (right for each shell element) and hence the frequency equation.

The sixteen homogenous simultaneous equations resulting from the substitution of Eq. (8) in Eq. (6) and then in the boundary conditions Eqs. (5) may be written as:

\[ \sum_{k=1}^{16} A_{kj}(\omega_i^2, \alpha_i, U), C_j = 0 \quad j=1,2, \ldots, 16 \]

\[ \text{...............(11)} \]

where \( A_{kj} \) are functions of the natural frequency \( \omega_i \), the fluid type \( \alpha_i \) and the fluid velocity \( U \), and are generated from the boundary conditions. For non-trivial solution of the set of equations (11) the determinant of the coefficients \( A_{kj} \)'s must vanish, \( |A_{kj}| = 0 \). This results in a characteristic equation whose eigenvalues determine the natural frequencies of the blade as a function of the blade proportions and the fluid parameters.

Natural frequencies of a specific blade with fixed flow condition are obtained by iteration process after setting the required axial mode number and the initial estimate of the natural frequency.

3. EXPERIMENTAL MODEL

Two models were prepared and tested using air and water as the flowing fluid. These models with the specifications given in Table (1) were constructed from aluminum sheets. For each model two sheets of the same thickness were rolled to the required radius, cut to the proper width and length and then joined in the manner shown in Fig. (5). It is felt that this will simulate the blade and insure the rigidity of the longitudinal boundaries.

To simulate the clamped boundary conditions for different model dimensions and to supply the model with the required testing fluid an adjustable testing rig, Fig. (6), fabricated from steel blocks was constructed. The rig was fastened together with the help of bolts and the water leakage was prevented using rubber gaskets at the interfaces between different parts of the rig. The model fixing end was formed into a rectangular shape using plastic-steel binder (Devicon type A) in order to fit in the rectangular opening of the rig.

Testing fluid was supplied from a compressor or a pump through a rotameter and a pressure gauge for fluid flow measurements. Model excitation was performed using a vibrator and a resonant condition was determined by holding a pickup in a stationary position while varying the frequency of oscillation. A natural frequency with and without fluid flow was distinguished by observing the sharp increase in amplitude of the displaced output signal and by the intensity of the acoustic tone emitted.

4. RESULTS AND DISCUSSION

During the course of the present investigation numerical computations were performed to evaluate the effect of the individual terms in Eq. (7b). This shows that the main effect of the fluid flow on the resonance frequencies is due to the equivalent fluid inertial term (the first term on the R.H.S of equation 7b) whereas the centrifugal term (the last term on the R.H.S...
of equation 7b) has a relatively smaller effect at moderate flow velocities. Therefore, the fluid density plays the major role in determining this effect. To clarify this, if air is considered as the flowing fluid where the density is small, the contribution of the two fluid affecting terms is very small on the value of the coefficient Bn, Eq. (7b), consequently, the effect of these terms on the system natural frequencies is negligible, Table (2). However, with water as the flowing fluid where the density is much higher than that of air, the value of the two terms (with the inertial term having the predominant effect) have a larger effect on the frequency range parameter (\(\Delta Bn - \Delta Cn\)) and hence on the system resonance frequencies.

Table 2 Effect of air flow at (30 m/sec) on the natural frequencies of the typical blade model II

<table>
<thead>
<tr>
<th>Mode No</th>
<th>Theoretical frequencies [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without flow</td>
</tr>
<tr>
<td>1,1</td>
<td>90.054</td>
</tr>
<tr>
<td>1,2</td>
<td>116.17</td>
</tr>
<tr>
<td>2,1</td>
<td>221.95</td>
</tr>
<tr>
<td>2,2</td>
<td>269.43</td>
</tr>
<tr>
<td>3,1</td>
<td>462.67</td>
</tr>
<tr>
<td>3,2</td>
<td>486.79</td>
</tr>
<tr>
<td>3,3</td>
<td>872.28</td>
</tr>
</tbody>
</table>

The main numerical results obtained in this investigation are presented in Tables (2) through (4). Table (2) clearly shows that the effect of air as the flowing fluid is insignificant. This was justified by experimental results which are not given in this table because the differences in frequencies were within the experimental error.

Tables (3) and (4) show sets for the theoretical and experimental resonance frequencies recorded for the two typical blade models with water as the flowing fluid. The experimental values were obtained for water filled models (water at very low velocity) which were noticed to give no recordable change when the water velocity is increased. It can be concluded from these results that the effect of the presence of water is to reduce all of the system vibration resonance frequencies. Such effect is expected and is due to the added mass of water to the mass of the system.

Increasing the flow velocity of either fluids (air or water) have an increasing effect on all of the resonance frequencies associated with the first axial mode (n=1) while all other frequencies of the higher axial modes (n > 1) decrease. Similar fluid flow effect is obtained with a beam approach used to simulate the blade, Fig. (1). That approach (19) shows the small order of the fluid flow effect on the system natural frequencies.

For qualitative comparison purpose the last columns of tables (3) and (4) give the natural frequencies of a dynamically equivalent beam to the blade model. Where

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Table 3 Theoretical and experimental resonance frequencies (Hz) of blade model I

<table>
<thead>
<tr>
<th>Mode No</th>
<th>Theoretical</th>
<th>Experimental</th>
<th>Least approximate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without fluid</td>
<td>Water filled</td>
<td>Without fluid</td>
</tr>
<tr>
<td>1,1</td>
<td>30.26</td>
<td>24.29</td>
<td>24.29</td>
</tr>
<tr>
<td>1,2</td>
<td>183.72</td>
<td>133.92</td>
<td>133.92</td>
</tr>
<tr>
<td>1,3</td>
<td>221.95</td>
<td>183.72</td>
<td>183.72</td>
</tr>
<tr>
<td>1,4</td>
<td>330.25</td>
<td>277.77</td>
<td>277.77</td>
</tr>
<tr>
<td>2,1</td>
<td>367.76</td>
<td>249.54</td>
<td>249.54</td>
</tr>
<tr>
<td>2,2</td>
<td>386.35</td>
<td>291.65</td>
<td>291.65</td>
</tr>
<tr>
<td>3,1</td>
<td>399.97</td>
<td>314.71</td>
<td>314.71</td>
</tr>
<tr>
<td>3,2</td>
<td>427.80</td>
<td>340.91</td>
<td>340.91</td>
</tr>
<tr>
<td>3,3</td>
<td>534.60</td>
<td>361.37</td>
<td>361.37</td>
</tr>
</tbody>
</table>

Table 4 Theoretical and experimental resonance frequencies (Hz) of blade model II

<table>
<thead>
<tr>
<th>Mode No</th>
<th>Theoretical</th>
<th>Experimental</th>
<th>Least approximate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without fluid</td>
<td>Water filled</td>
<td>Without fluid</td>
</tr>
<tr>
<td>1,1</td>
<td>60.24</td>
<td>45.02</td>
<td>45.02</td>
</tr>
<tr>
<td>1,2</td>
<td>108.16</td>
<td>88.65</td>
<td>88.65</td>
</tr>
<tr>
<td>2,1</td>
<td>222.10</td>
<td>189.65</td>
<td>189.65</td>
</tr>
<tr>
<td>2,2</td>
<td>277.77</td>
<td>230.77</td>
<td>230.77</td>
</tr>
<tr>
<td>3,1</td>
<td>430.40</td>
<td>374.74</td>
<td>374.74</td>
</tr>
<tr>
<td>3,2</td>
<td>497.42</td>
<td>425.92</td>
<td>425.92</td>
</tr>
<tr>
<td>3,3</td>
<td>663.79</td>
<td>591.39</td>
<td>591.39</td>
</tr>
</tbody>
</table>

(1) Experimental frequency is not clearly identified because of interference with 360 Hz frequency.

(2) Frequency is not estimated by the particular method.

(4) Other frequencies were measured but not presented in these tables since the approximate theory used in the investigation does not predict all frequencies as normally determined by using exact theories.
the frequencies are calculated from the formula:

\[ \omega_n = \sqrt{\frac{f_n^2}{L}} \cdot \left( \sqrt{\frac{L}{A}} \right) \]

with \( A \) and \( I \) indicating the equivalent cross sectional area and moment of inertia of the composite blade section, Fig.(3), respectively.

The discrepancies between the predicted and measured frequencies are within acceptable range for the systems tested. These discrepancies may be due to the non-linearity of the actual models which is not being considered by the theory. Experimental error, model imperfections, fixing condition and the method of connection for the common edge of the two shell elements may play an important role of these discrepancies.

5. CONCLUSIONS

The shell model approach presented in this paper represents an excellent tool for estimating the effect of the coolant fluid on the natural frequencies at least in the preliminary stage of the design procedure.

For the assumed flow conditions, the fluid is seen to acts through its inertial, centrifugal and Coriolis forces. The inertial force has the predominant effect and always tends to reduce all of the system natural frequencies. The centrifugal fluid force which is proportional to the square of the flow velocity has the increasing effect on the first axial mode frequencies and a decreasing effect on the higher mode frequencies, however, this effect is not as much as the inertial effect. On the other hand the Coriolis fluid force is responsible for the system instability. Roughly speaking this force is proportional to the flow velocity and hence is expected to have a lower effect on the system natural frequencies (real part of the circular frequency) than that of the centrifugal effect.

REFERENCES


APPENDICES

1. The equilibrium equations which lead to equation (1) may be written as follows (11), Fig.(7):

x-direction \[ \frac{R}{1} \frac{\partial N_x}{\partial x} + \frac{\partial N_x}{\partial \xi} = 0 \] (12)

y-direction \[ \frac{R}{1} \frac{\partial N_y}{\partial x} + \frac{\partial N_y}{\partial \xi} = 0 \] (13)

z-direction \[ \frac{1}{R} \frac{\partial M_z}{\partial x} + N + F \rho \omega R \frac{\partial^2 \xi}{\partial t^2} = 0 \] (14)

Combining the above equations results in Eq.(1) and the following functional relations:

\[ \omega = \sqrt{\frac{12 \frac{R^3}{E} \frac{\partial^2 \xi}{\partial t^2}}{12 \frac{R^3}{E} \frac{\partial^2 \xi}{\partial t^2}}} \] (19)

\[ v = - \frac{12 \frac{R^3}{E} \frac{\partial^2 \xi}{\partial t^2}}{3 \frac{R^3}{E} \frac{\partial^2 \xi}{\partial t^2}} \] (20)

\[ w = - \frac{12 \frac{R^3}{E} \frac{\partial^2 \xi}{\partial t^2}}{3 \frac{R^3}{E} \frac{\partial^2 \xi}{\partial t^2}} \] (21)

Fig. 7 Forces and moments on a cylindrical shell element.
The dynamic fluid pressure $P_d$, Eq. (2) acting on a shell element may be derived from:

$$ P_d = \frac{d}{dr} \left[ \frac{R^2}{\rho} \frac{\partial^2 \Phi}{\partial t^2} \right] - \frac{\partial}{\partial r} \left[ \frac{R^2}{\rho} \frac{\partial \Phi}{\partial r} \right] - \frac{\partial^2}{\partial \theta^2} \Phi \quad \text{....(27)} $$

where $\Phi$ is the perturbation velocity potential satisfying in cylindrical coordinate the wave equation:

$$ \nabla^2 \Phi = \frac{1}{\rho} \left( \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial}{\partial r} \left( \rho \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial \theta^2} \right) \quad \text{....(28)} $$

It is assumed that the shell wall and the fluid remain in contact and so,

$$ \frac{1}{r} \frac{d}{dr} \left[ \frac{R^2}{\rho} \frac{\partial \Phi}{\partial r} \right] - \frac{\partial \Phi}{\partial t} = - \frac{\partial \Phi}{\partial t} + \frac{\partial^2 \Phi}{\partial \theta^2} \quad \text{....(29)} $$

Supposing the function $\Phi$ to be separable in the following way:

$$ \Phi(r,\theta,\tau) = \tilde{\Phi}(r)\Phi(\theta,\tau) \quad \text{....(30)} $$

then by using the boundary condition (29), the function $\Phi$ may be written as:

$$ \Phi = \frac{R \tilde{\Phi}(r)}{\tilde{\Phi}(r)} \left( \frac{\partial \Phi}{\partial t} + U \frac{\partial \Phi}{\partial \theta} \right) \quad \text{....(31)} $$

and thus the dynamic pressure $P_d$ is found to be:

$$ P_d = \frac{R \tilde{\Phi}(r)}{\tilde{\Phi}(r)} \left( \frac{\partial \Phi}{\partial t} + U \frac{\partial \Phi}{\partial \theta} + \frac{\partial^2 \Phi}{\partial \theta^2} \right) \quad \text{....(32)} $$

Substituting for $\Phi$, Eq. (21) in Eq. (32) results in the required pressure equation of the form (2).

An expression similar to Eq. (32) was obtained for a beam element which gave equation (9) is found to be, Ref. [9]:

$$ P_d = \frac{A^2}{\rho} \left( \frac{\partial^2 \Phi}{\partial t^2} + 2U \frac{\partial \Phi}{\partial \theta} + U^2 \frac{\partial^2 \Phi}{\partial \theta^2} - \frac{\partial^2 \Phi}{\partial \theta^2} \right) \quad \text{....(33)} $$

Equation (33) represents the rate of change of momentum per unit length and width of a fluid element flowing in a beam element of a flow area $A$ and width $b$.

3. The axial mode function $X_n$ in Eq. (6b) may be written as follows:

$$ X_n = \cosh(\lambda_n \xi) - \cos(\lambda_n \xi) - \sigma_n (\sinh(\lambda_n \xi) - \sin(\lambda_n \xi)) \quad \text{....(34)} $$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$\lambda_n$ & $\sigma_n$ & $X_n$ \tabularnewline
\hline
1 & $\cosh(\lambda_1)$ & $\cosh(\lambda_1)$ \tabularnewline
2 & $\cosh(\lambda_2)$ & $\cosh(\lambda_2)$ \tabularnewline
3 & $\cosh(\lambda_3)$ & $\cosh(\lambda_3)$ \tabularnewline
4 & $\cosh(\lambda_4)$ & $\cosh(\lambda_4)$ \tabularnewline
5 & $\cosh(\lambda_5)$ & $\cosh(\lambda_5)$ \tabularnewline
6 & $\cosh(\lambda_6)$ & $\cosh(\lambda_6)$ \tabularnewline
7 & $\cosh(\lambda_7)$ & $\cosh(\lambda_7)$ \tabularnewline
8 & $\cosh(\lambda_8)$ & $\cosh(\lambda_8)$ \tabularnewline
\hline
\end{tabular}
\caption{Values of the function $Y_{mn}$}
\end{table}